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Viscoplasticity and the dynamics of brittle fracture

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I propose a model of fracture in which the curvature of the crack tip is a relevant dynamical variable and crack advance is governed solely by plastic deformation of the material near the tip. This model is based on a rate-and-state theory of plasticity introduced in earlier papers by Falk, Lobkovsky, and myself. In the approximate analysis developed here, fracture is brittle whenever the plastic yield stress is nonzero. The tip curvature finds a stable steady-state value at all loading strengths, and the tip stress remains at or near the plastic yield stress. The crack speed grows linearly with the square of the effective stress intensity factor above a threshold that depends on the surface tension. This result provides a possible answer to the fundamental question of how breaking stresses are transmitted through plastic zones near crack tips.

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I. INTRODUCTION

Among the most intriguing puzzles in nonequilibrium materials physics is the question of how brittle fracture can occur in viscoplastic solids. Simply stated, most solids flow plastically at some yield stress s_y and therefore cannot support steady stresses larger than this value. However, s_y is ordinarily assumed to be smaller than the cohesive stress needed to break bonds at the tip of a crack. How then can the breaking stress be transmitted through the plastic zone near the tip? This question reveals a deep inconsistency between conventional theories of plasticity on the one hand [1,2] and conventional descriptions of brittle fracture on the other [3,4].

One recent attempt to solve the tip-stress puzzle is the strain-gradient theory of Fleck *et al.* [5–7]. The idea here is that the tangle of dislocations made geometrically necessary by the strong strain gradients at the crack tip hardens the material in that region, permitting large stresses to reach the tip. This picture cannot be entirely satisfactory; the same phenomena occur in noncrystalline materials where the dislocation mechanism is not relevant. Moreover, the geometric necessity of dislocations in a deformed material depends on the assumption that crystalline order persists and bends with the overall deformation, instead of dissolving and reforming in new directions. It is not clear (to me) which of these pictures is most accurate in the strong stress field near a moving crack tip.

Some years ago, Freund and Hutchinson [8] (FH) pursued a different line of argument. They used the fact that, in viscoplastic materials, (deviatoric) stresses above s_y drive plastic strain *rates* (not just strains) which, although not sustainable in purely stationary situations, cause only finite deformations in the transitory neighborhood of a moving crack tip. The FH calculation consists simply of taking the well-known formula for the stresses in the neighborhood of a geometrically sharp crack whose tip is moving at speed v_{tip}, computing from this the plastic strain rate according to a phenomenological law of viscoplasticity and, finally, using this strain rate to compute the dissipative part of the energy release rate $G(v_{tip})$.

The result of any calculation of this kind necessarily must have the property that $G(v_{tip})$ diverges at small v_{tip}. In a static or nearly static stress field like that at the tip of a slow crack, the viscoplastic energy dissipation occurs at a fixed rate per unit time. Therefore $G(v_{tip})$, the elastic energy released per unit crack extension, must be proportional to $1/v_{tip}$ as v_{tip} approaches zero, a manifestly unstable behavior. One missing ingredient in the Freund-Hutchinson calculation is the shape of the crack tip. Like most theorists in this field, these authors assumed a geometrically sharp crack with its associated stress singularity, and did not allow their crack tip to blunt in such a way as to regularize that singularity. It is hard to understand, however, how a crack tip in a deformable, viscoplastic material can remain infinitely sharp.

My dissatisfaction with sharp-tip models of fracture has been intensified recently by my attempt, in collaboration with Lobkovsky [9], to use the cohesive-zone models of Barenblatt [10] and Dugdale [11] for studying the dynamic response of cracks to bending forces. We concluded that these models are mathematically and/or physically ill posed. The trouble is that the elastic stresses near a sharp tip, even when regularized by cohesive stresses, have features that are physically unreasonable. The sharp tip is too strong a constraint to be consistent with the physical forces acting in its neighborhood.

There is yet one more bit of evidence that is relevant. In an earlier paper on the cohesive-zone model, Ching, Nakanishi, and I [12] pointed out that the stresses near the sharp tip of a moving crack always have the property that the tangential stress is bigger than the normal (opening) stress. In the present context, this inequality means that the stress is always such that it would cause the tip to become blunt if the sharp-tip constraint were removed. Xu, Needleman, and Abraham [13] have seen just such an effect in their numerical studies using a cohesive-surface scheme in which blunting seems to be approximated by tip splitting.

With these considerations in mind, I have explored the possibility that the curvature of the crack tip is a relevant dynamical variable in brittle fracture. My purpose in this paper is to show, via a series of approximate calculations, that the tip-stress puzzle disappears if I assume that the motion of a crack is governed solely by plastic deformation of the material near its tip. Like Dugdale [11], I propose that the cohesive stress is essentially the same as the plastic yield stress. In contrast to the cohesive-zone models [10,11], however, the concentrated stress in this theory is regularized by tip blunting instead of by the length of the zone. That is, the radius of curvature of the tip is determined by some mechanism that resembles plastic flow in its immediate neighborhood. As I shall show, there is a simple, zero-velocity threshold for crack motion, and there is no hint of a low-speed instability.

In presenting this result, I first shall argue that conventional treatments of plasticity are inadequate for describing time-dependent, spatially inhomogeneous deformations near crack tips. Conventional theories couched in terms of yield criteria, with sharp distinctions between time-dependent and time-independent properties [1,2,4], cannot describe in any natural way the wide range of behaviors that occur in deformable solids as they undergo large, rapidly varying stresses. Plastic deformation is an inherently dynamic phenomenon. Strain hardening, the transition between viscoelastic and viscoplastic behavior at the yield stress, hysteretic stress-strain relations, etc., are all dynamic responses to applied forces. They all should be determined by constitutive equations of motion for some set of variables that describe the state of the system, both its shape (the displacement field) and its relevant internal characteristics.

I develop this argument in Sec. II by briefly summarizing the shear-transformation-zone (STZ) theory that Falk and I introduced in an analysis of plastic deformation in an amorphous solid [14]. The STZ theory is a specific example of a fully dynamic description of plasticity consistent with my remarks in the preceding paragraph. It is a ''rate-and-state'' theory (a term used widely in the seismological literature and in recent theories of friction [15–17]) that contains physically motivated internal state variables. Although Ref. [14] deals entirely with a simple model of a noncrystalline material, I believe that its main conclusions apply also to crystalline solids when the mean free paths of the mobile defects—impurities, vacancies, dislocations, etc.—are short compared to the length scales over which deformation and failure are occurring.

Section III, then, is based on work by Lobkovsky and myself [18] in which we compared predictions of the STZ theory with those of some conventional analyses in a spatially nonuniform situation. My main purpose in introducing the latter results is that they let me explore an approximation that I need for the fracture problem. As I shall emphasize, that approximation is not valid in conventional descriptions of plasticity. Lobkovsky and I considered a ''toy'' problem, specifically, an expanding circular hole in a very large, flat plate with outward tractions (negative pressure) at infinity. We found a threshold stress above which the hole grows without bound (a cavitation instability), and we suggested that this dynamic behavior might resemble plastic flow near a crack tip. At stresses below this threshold, we found that the STZ theory produces a time-independent plastic zone near the hole, but that this zone has conventional properties only in a limit in which the STZ model becomes perfectly plastic. For STZ parameters that correspond to appreciable strain hardening, we found thinner and smoother plastic zones both below the threshold and in the flowing region above it. It is this picture that I shall invoke in my analysis of the fracture problem.

In Secs. IV and V, I show how the tip-stress puzzle may be resolved in a dynamic picture of plastic tip blunting. I consider only slow, mode-I fracture. Rather than solve a complete free-boundary problem for the motion of a blunted crack, the best I have been able to do so far—in the spirit of the circular calculations in Sec. III—is to look at a long, thin, elliptical hole in a very large plate, with a uniaxial stress applied at infinity perpendicular to the long axis of the hole. Like Griffith [19], I take the limit in which the ratio of the short axis to the long axis goes to zero, but here I insist that the curvature at the sharp end, i.e., at the crack tip, remain finite. For this geometry, I can compute the stress field in the absence of plastic deformation and, from this field, using the approximation developed in Sec. III, estimate the plastic strain rate. Finally, I compute the velocity of the elliptical boundary, i.e., the rate at which the sharp end of the ellipse is advancing and the rate at which its curvature is changing. If I make the additional strong assumption that the crack remains everywhere elliptical throughout this deformation, then I have a closed set of equations of motion for the displacement and curvature at the crack tip. The result is a remarkably simple and compact description of brittle fracture in a deformable material that seems to avoid the internal inconsistency of conventional theories.

I conclude in Sec. VI with some remarks about various aspects of the STZ theory and fracture dynamics.

II. STZ MODEL

In Ref. [14], Falk and I described both molecular-dynamics simulations and theoretical analyses of pure shear deformations in a two-component, two-dimensional, noncrystalline, Lennard-Jones solid. The simulations revealed a rich variety of behaviors typical of real solids including viscoelasticity, viscoplasticity, strain hardening, and hysteresis. An especially important aspect of these simulations was that we could look inside the system to see in detail the irreversible molecular rearrangements that occur during plastic deformation. We found that these rearrangements are localized—that small regions which we called ''shear transformation zones'' deform in the direction of the applied stress. Once deformed, these regions are deactivated; i.e., they are ''jammed'' and cannot deform further in the same direction, but they can return to their previous orientations when the stress is reversed.

This two-state nature of the STZ's explains both the memory effects and the reason why plastic deformations are limited in size for deviatoric stresses less than a yield stress s_y. (The deviatoric yield stress s_y is half the more familiar uniaxial yield stress.) In an extension of this idea, we interpreted the change in behavior near s_y as the result of creation and annihilation of STZ's at rates proportional to the rate at which plastic work is done on the material. When active new STZ's are created as fast as existing ones are deactivated, the material continues to deform indefinitely under constant stress.

Rather than discuss the detailed theoretical interpretation of the numerical experiments developed in [14], I present here only a truncated version of the STZ theory that omits a strongly s-dependent rate factor that governs memory effects. This truncation is sensible only if we load the system just once in one direction, but it does not behave properly if the loading is cycled. It will be good enough for present purposes but not for all aspects of the fracture problem. (See remarks in Sec. VI.)

Lobkovsky and I used this approximation in our analysis of the circle problem [18]. We wrote

$$\dot{\varepsilon}^{pl} \approx \frac{1}{\tau}(\lambda s - \Delta), \qquad (2.1)$$

where $\dot{\varepsilon}^{pl}$ is the plastic strain rate (more accurately, the plastic rate-of-deformation tensor), s is the deviatoric stress, and Δ is a tensor that describes the anisotropy in the orientations of the STZ's. Note that the strain rate is reduced by Δ. Larger values of Δ indicate that larger numbers of the STZ's have switched into the direction of the shearing deformation and can no longer undergo forward transitions. When Δ reaches the value λs, the system becomes ''jammed'' and the strain rate vanishes. The parameter τ simply sets a time scale for the system.

Our equation of motion for Δ is

$$\dot{\Delta} \approx \dot{\varepsilon}^{pl} - \left(\frac{\dot{\varepsilon}^{pl} \cdot s}{\lambda s_y^2} \right) \Delta. \qquad (2.2)$$

The first term on the left-hand side expresses the fact that the rate at which the STZ's are switching is the same as the plastic strain rate and therefore, in the absence of annihilation and creation of the STZ's, Δ grows at the rate $\dot{\varepsilon}^{pl}$. Annihilation and creation are described by the second term, proportional to the rate of plastic work $(\dot{\varepsilon}^{pl} \cdot s)$.

To understand what is happening here, consider a spatially uniform system. For s less than the yield stress s_y, $\Delta(t)$ has its stable fixed points on the line $\Delta = \lambda s$, where both $\dot{\varepsilon}^{pl}$ and $\dot{\Delta}$ vanish. In contrast, for $s > s_y$, the stable fixed points of Δ are on the line $\Delta = \lambda s_y^2/s$, where the strain rate $\dot{\varepsilon}^{pl}$ does not vanish. In the region $s < s_y$, the material is viscoelastic; that is, it deforms but does not flow indefinitely in response to the applied stress. For $s \ll s_y$, we can neglect the nonlinear second term in Eq. (2.2) and set $\Delta \approx \varepsilon^{pl}$. Then Eq. (2.1) becomes

$$\dot{\varepsilon}^{pl} + \frac{1}{\tau}\varepsilon^{pl} \approx \frac{\lambda}{\tau} s, \qquad (2.3)$$

which is equivalent to a simple creep-compliance law with relaxation time τ.

For larger values of s, but still for $s < s_y$, the relation between ε^{pl} and s becomes nonlinear as well as noninstantaneous. Equations (2.1) and (2.2) can be integrated to yield

$$\varepsilon^{pl}_{final} \equiv \varepsilon^{pl}(t \to \infty) = -\frac{\lambda s_y^2}{s} \ln\left(1 - \frac{s^2}{s_y^2}\right). \qquad (2.4)$$

The conditions leading to Eq. (2.4) are that the system be initially in a state with $\varepsilon^{pl} = \Delta = 0$, that a stress $s < s_y$ be applied suddenly at time $t = 0$, and that the system reach a final state where $\Delta_{final} = \lambda s$, $\dot{\Delta} = 0$. A further calculation then tells us that $\varepsilon^{pl}(t)$ approaches ε^{pl}_{final} exponentially (like $\exp[-t/\tau_{relax}]$) with a relaxation time τ_{relax} that diverges as $s \to s_y$:

$$\tau_{relax} = \frac{\tau}{1 - (s/s_y)^2}. \qquad (2.5)$$

Equation (2.4) is a strain-hardening curve; that is, ε^{pl}_{final} is the plastic strain produced after an infinitely long time by the deviatoric stress s. In the limit $\lambda \to 0$, with s_y held constant, ε^{pl}_{final} vanishes for $s < s_y$ but can have any positive value for $s = s_y$; that is, the STZ model becomes ''perfectly plastic'' in this limit. The parameter λ is a measure of the deviation from this ideal behavior. The diverging relaxation time near $s = s_y$ has no analog in conventional descriptions of strain hardening. Indeed, this interpretation of the hardening curve is different from that of conventional theories in which such curves are treated as instantaneous, nonlinear response functions.

Finally, note that, for $s > s_y$, $\Delta \to \lambda s_y^2/s$,

$$\dot{\varepsilon}^{pl} \to \frac{\lambda}{\tau s}(s^2 - s_y^2) \cong \begin{cases} \dfrac{2\lambda}{\tau}(s - s_y) & \text{for } s - s_y \ll s_y, \\ \dfrac{\lambda}{\tau}(s - s_y) & \text{for } s \gg s_y. \end{cases} \qquad (2.6)$$

Apart from the factor of 2 difference between the small-s and large-s behavior, this is a ''Bingham plastic.'' In dynamic situations at large stress, therefore, the STZ model exhibits very nearly conventional viscoplastic behavior.

In short, even this highly truncated version of the STZ theory provides a compact and physically motivated description of much of plasticity theory, both static and time dependent. In just two constitutive relations, Eqs. (2.1) and (2.2), we capture linear viscoelasticity at small stress, strain hardening at larger stress, and a dynamic transition to viscoplasticity at a yield stress s_y.

III. EXPANDING CIRCULAR HOLE

To make contact between the STZ theory and conventional theories of plasticity, and also to gain insight that might be useful in the crack-tip problem, Lobkovsky and I [18] studied the quasistatic (noninertial) dynamics of a circular hole in a large plate with outward traction (pressure $p \to -\sigma_\infty$) at infinity. We assumed plane strain, neglected surface tension at the boundary of the hole, and, in most of our calculations, assumed incompressible elasticity (Poisson's ratio $\nu = 1/2$). In the case of circular symmetry, with polar coordinates r and ϕ, we used the STZ equations (2.1) and (2.2) with the deviatoric stress s being defined as $s = s_{\phi\phi}(r) = -s_{rr}(r)$. Similarly, $\Delta(r) = \Delta_{\phi\phi}(r) = -\Delta_{rr}(r)$. Our results were as follows.

When the applied stress at infinity, σ_∞, is not too much greater than s_y, the STZ model is consistent with conventional time-independent plasticity theory. That is, for small values of the dimensionless quantity λs_y, a well-defined plastic zone forms around the hole. Within that zone, $s \cong s_y$ and $\Delta \cong \lambda s_y$. The only difference from most conventional results is that the function $s(r)$, where r is the radial distance from the center of the hole, makes a smooth transition from $s \cong s_y$ inside the plastic zone to $s \sim 1/r^2$ outside—a behavior that can be recovered in some strain-hardening theories. The transition becomes sharp in the limit of perfect plasticity, $\lambda \to 0$.

As in conventional time-independent theories, the equilibrium radius of the hole diverges at a threshold stress

$$\sigma_\infty^{th} \approx \begin{cases} s_y - s_y \ln(2\lambda s_y + s_y/\mu) & \text{for } 2\lambda s_y + s_y/\mu \ll 1, \\ 1/2\lambda & \text{for } \lambda s_y \gg 1, \end{cases} \tag{3.1}$$

where μ is the shear modulus and $\nu = 1/2$. Again, this agrees with the conventional result in the limit $\lambda \to 0$. For applied stresses just slightly larger than σ_∞^{th}, that is, $\sigma_\infty - \sigma_\infty^{th} \ll \sigma_\infty^{th}$, the radius of the hole $R(t)$ grows exponentially at the rate

$$\frac{\dot{R}}{R} \approx \begin{cases} (2\lambda/\tau)[1 + 2\lambda s_y \ln(\lambda s_y)](\sigma_\infty - \sigma_\infty^{th}) & \text{for } s_y/\mu \ll \lambda s_y \ll 1, \\ (\lambda/\tau)(\sigma_\infty - \sigma_\infty^{th}) & \text{for } s_y/\mu \ll 1 \ll \lambda s_y. \end{cases} \tag{3.2}$$

In each of the results shown in Eq. (3.2), the quantity σ_∞^{th} has the value given in the corresponding part of Eq. (3.1). Using methods similar to those used to derive Eq. (3.2), I find, for $s_y/\mu \ll \lambda s_y \ll 1$ and for very large σ_∞,

$$\frac{\dot{R}}{R} \approx \frac{\lambda}{\tau}(\sigma_\infty - s_y). \tag{3.3}$$

An especially important aspect of these results is that, as long as we retain nonzero values of λ, we can work in the limit $\mu \to \infty$. That is, at least for exploratory purposes, we can neglect elastic displacements in using the STZ model. The relevant dimensionless group of parameters in the STZ analysis is s_y/μ, which is of order 0.1 or less for many real materials. Falk and I [14] found $s_y/\mu \cong 0.03$ for the two-dimensional noncrystalline material that we used in our numerical experiments. Thus this theoretical limit, which I shall use here primarily for analytic convenience, may be physically realistic.

Unlike the STZ analysis, conventional theories effectively set $\lambda = 0$ at the beginning of the calculation. They then find, as seen in Eq. (3.1), that $\sigma_\infty^{th} \to \infty$ as $\mu \to \infty$. Moreover, for values of σ_∞ just below threshold, conventional theories predict that the ratio of the radius of the outer boundary of the plastic zone, R_1, to the radius of the hole, R, is of order $\sqrt{\mu/s_y}$, which also diverges in the large-μ limit. The reason for this behavior of the conventional theories is that they typically allow no plastic deformation outside the plastic zone; thus, the outer displacements required by compatibility must be elastic. If those displacements are constrained by the stiffness of the material, then growth of the hole by plastic flow must likewise be constrained.

The STZ theory is quite different in this regard. According to Eq. (3.1), the large-μ threshold remains approximately at s_y except for either very small or very large values of λs_y. For near-threshold values of σ_∞ and $s_y/\mu \to 0$, we have [18]

$$s(r) = s_y \tanh\left(\frac{R^2}{2\lambda s_y r^2}\right), \tag{3.4}$$

which implies that $R_1/R \cong 1/\sqrt{2\lambda s_y}$. Thus the STZ theory predicts a smooth and relatively thin plastic zone near the surface of the hole.

With this picture in mind, we can begin to think about a simple approximation for hole growth that might be useful in the fracture problem. Note that, in the purely elastic (nonplastic) version of the hole problem, force balance and compatibility, plus the condition that the normal stress vanish at the edge of the hole, imply that the deviatoric stress is

$$s(r) = \sigma_\infty R^2/r^2. \tag{3.5}$$

The shear modulus μ does not occur here; this formula remains valid in the limit of infinite elastic stiffness. Near threshold, where σ_∞ is of order s_y, Eq. (3.5) is at least qualitatively correct according to Eq. (3.4) for values of λs_y roughly of order unity. Far above threshold, where we are well into the flowing regime described by Eq. (2.6), Δ is small and Eq. (3.5) becomes quantitatively accurate.

Now let us try to estimate the rate of plastic deformation induced by this stress field at $r = R$. As is well known, this is not a well-posed problem. For any constitutive relation between stress and plastic deformation rate (except a strictly linear relation with no yield stress), the stress tensor associated with Eq. (3.5) is not generally compatible with the vector velocity field that we are trying to compute. We can minimize (but not entirely remove) this difficulty by, first, using a stress field that is reasonably accurate as argued above and, second, by using a local form of the constitutive relation. Specifically, start with

$$\frac{\dot{R}}{R} = \dot{\varepsilon}^{pl}_{\phi\phi}(R) = -\dot{\varepsilon}^{pl}_{rr}(R) = \mathcal{D}[s(R)], \tag{3.6}$$

which is an exact formula in this circularly symmetric geometry for incompressible plasticity and for $\mu \to \infty$. Here, $\mathcal{D}$ is an approximate constitutive law, relating deviatoric stress and the rate of plastic deformation, which must be as simple as possible in order to be analytically useful.

If we choose

$$\mathcal{D}(s) = \begin{cases} (\lambda/\tau)(s - s_y) & \text{for } s > s_y, \\ 0 & \text{otherwise}, \end{cases} \tag{3.7}$$

then, using Eq. (3.5), we immediately find

$$\frac{\dot{R}}{R} \cong \frac{\lambda}{\tau}(\sigma_\infty - s_y). \tag{3.8}$$

This is exactly the same as Eq. (3.3) for large σ_∞, as might have been expected because Eq. (3.5) is also correct in that limit. For values of σ_∞ near threshold, Eq. (3.8) is qualitatively sensible. The threshold is approximately correct for values of λs_y of order unity, and the growth rate vanishes linearly in $(\sigma_\infty - s_y)$ with slope λ/τ. For small values of λs_y, where the STZ plastic zone is more extended and the deviatoric stress at the surface of the hole is substantially less than σ_∞, Eq. (3.8) underestimates the threshold and overestimates the growth rate.

There are other schemes that do not work so well. For example, instead of starting with Eq. (3.6), we can use the relation between the radial velocity v_r and the rate of deformation $\mathcal{D}$: $dv_r/dr = \dot{\varepsilon}^{pl}_{rr} = -\mathcal{D}(s(r))$, and then compute $v_r(R) = \dot{R}$ by integrating this relation from $r = R$ out to the edge of the plastic zone where $\mathcal{D}$ vanishes. The result is an expression for $\dot{R}$ that vanishes quadratically, like $(\sigma_\infty - s_y)^2$ near threshold, and also deviates substantially from Eq. (3.8) at large σ_∞. These qualitative discrepancies indicate violations of compatibility; they would not occur if the stress (3.5) were exactly consistent with the constitutive relation (3.7), and they disappear if I regain compatibility by setting $s_y = 0$. The approximation based on the local formula (3.6) works well, apparently, because it is not so sensitive to the compatibility violations or to the fact that Eq. (3.7) is not an accurate version of the STZ model near the outer edge of the plastic zone.

IV. ELLIPTICAL APPROXIMATION FOR CRACK-TIP DYNAMICS: MATHEMATICAL PRELIMINARIES

My strategy now is to use the approximations developed in the preceding section in an analysis of a highly elongated elliptical—rather than circular—hole, and in this way to explore the effects of tip blunting in fracture dynamics as outlined in the Introduction. As mentioned there, the idea is to compute the instantaneous rate of deformation of the ellipse due to plastic displacements and then to assume that the deforming hole remains elliptical in order to compute subsequent motion.

The first step is to compute the stresses in the neighborhood of the hole, in analogy to the (much shorter) calculation that leads to Eq. (3.5). The elliptical version of this calculation, for the case of zero surface tension, can be found in Muskhelishvili [20]. We need Muskhelishvili's results for the case in which the stress infinitely far from the hole, σ_∞ (''p'' in Muskhelishvili's notation) is oriented along the y axis in the x,y plane.

To start, make the conformal transformation from Cartesian coordinates (x,y) to elliptical coordinates (ρ,θ):

$$x = W\left(\rho + \frac{m}{\rho}\right)\cos\theta, \quad y = W\left(\rho - \frac{m}{\rho}\right)\sin\theta. \tag{4.1}$$

Curves of constant ρ are ellipses, and curves of constant θ are orthogonal hyperbolas. If we take the boundary of the elliptical hole to be at $\rho = 1$, then the semimajor and semiminor axes of the ellipse have lengths $W(1+m)$ and $W(1-m)$, respectively. We take $0 < m < 1$ so that the long axis of the ellipse lies in the x direction, perpendicular to the applied stress, in analogy to a mode-I crack.

These elliptical coordinates provide an orthogonal basis for a representation of the stress tensor σ. Mushkelishvili's results are

$$\sigma_{\rho\rho} + \sigma_{\theta\theta} = \sigma_\infty \mathrm{Re}\left[1 + \frac{2(1+m)e^{-2i\theta}}{\rho^2 - me^{-2i\theta}}\right] \tag{4.2}$$

and

$$\mathcal{S}(\rho,\theta) \equiv \sigma_{\theta\theta} - \sigma_{\rho\rho} + 2i\sigma_{\rho\theta} = \frac{\sigma_\infty \rho^2 e^{2i\theta}}{(\rho^2 - me^{2i\theta})}\left[1 - \frac{e^{-2i\theta}}{m\rho^2} + \frac{(1+m)e^{-2i\theta}}{(\rho^2 - me^{-2i\theta})^2} M(\rho,\theta)\right], \tag{4.3}$$

where

$$M(\rho,\theta) = \frac{\rho^2}{m}(1 - 2me^{-2i\theta} + m^2) + e^{-2i\theta}(1 - 2me^{2i\theta} + m^2). \tag{4.4}$$

The deviatoric stress has components

$$s_{\theta\theta} = -s_{\rho\rho} = \frac{1}{2}\mathrm{Re}\,\mathcal{S}(\rho,\theta), \quad s_{\rho\theta} = \frac{1}{2}\mathrm{Im}\,\mathcal{S}(\rho,\theta). \tag{4.5}$$

To produce a long, thin ellipse, let W become larger than any other length scale in the system, and fix $m \leqslant 1$ so that the curvature $\mathcal{K}_{tip}$ at the tip, that is, at $x = W(1+m)$, remains finite. Then a calculation to leading order in $W^{-1/2}$ yields

$$m \approx 1 - \sqrt{\frac{2}{\mathcal{K}_{tip} W}}. \tag{4.6}$$

Throughout what follows, the symbol $\approx$ denotes the large-W limit.

We can see in more detail what is happening near this crack tip by looking at the stress along the x axis. Set $\theta = 0$ in Eq. (4.1) and solve for ρ as a function of the distance from the tip, $\tilde{x} = x - W(1+m)$. The result is

$$\rho(\theta = 0) \approx 1 + \frac{1}{\sqrt{W}}\left[\left(\tilde{x} + \frac{1}{2\mathcal{K}_{tip}}\right)^{1/2} - \left(\frac{1}{2\mathcal{K}_{tip}}\right)^{1/2}\right]. \tag{4.7}$$

For very large W and for $\tilde{x} \ll W$, Eq. (4.2) and (4.3) produce

$$\sigma_{yy}(\tilde{x}, y=0) \approx \frac{3\sigma_\infty\sqrt{W}}{2(\tilde{x} + 1/2\mathcal{K}_{tip})^{1/2}} + \frac{\sigma_\infty\sqrt{W}}{2\mathcal{K}_{tip}(\tilde{x} + 1/2\mathcal{K}_{tip})^{3/2}}. \tag{4.8}$$

Only the first term on the right-hand side of Eq. (4.8) contributes to the asymptotic behavior for $\mathcal{K}_{tip}\tilde{x} \gg 1$; therefore the mode-I stress-intensity factor must be $K_I = (3/2)\sigma_\infty\sqrt{\pi W}$. When we take the limit of large W, we must let σ_∞ become small so that K_I remains fixed.

In order to determine the motion of the elliptical crack tip, we need to compute $v_n(\theta)$, the normal velocity of the material on the surface $\rho=1$, and the rate of change of the curvature of this surface, $\dot{\mathcal{K}}(\theta)$, near the tip, that is, near $\theta=0$. There is a simple relation between v_n and $\dot{\mathcal{K}}$, valid for any curve [21]:

$$\dot{\mathcal{K}}=-\mathcal{K}^2 v_n-\frac{\partial^2 v_n}{\partial w^2}, \tag{4.9}$$

where w is the displacement along the curve, and $\mathcal{K}$ and v_n are chosen to be positive when the curve bends to the left and v_n points to the right as w increases. The relation between w and the elliptical coordinate θ is

$$\frac{1}{W^2}\left(\frac{dw}{d\theta}\right)^2=\rho^2\left(1+\frac{m^2}{\rho^4}\right)-2m\cos 2\theta\approx\frac{2}{\mathcal{K}_{tip}W}+4\theta^2, \tag{4.10}$$

the final expression being valid for $\rho=1$ and large W. Combining the last two results, we find

$$-\frac{\dot{\mathcal{K}}_{tip}}{\mathcal{K}_{tip}^2}=v_n(0)+\frac{1}{2\mathcal{K}_{tip}W}\left.\frac{\partial^2 v_n}{\partial\theta^2}\right|_{\theta=0}. \tag{4.11}$$

The quantity $v_n(0)$ is equal to v_{tip}, the speed at which the tip of the crack is advancing.

From these last several equations, we see that θ is small of order $W^{-1/2}$ for values of $\mathcal{K}_{tip}w$ of order unity, which is the region of interest. Thus, we need to keep θ in our equations only when it occurs in the combination $\sqrt{W}\theta$, and we need to keep only terms up to order θ^2 for computing $\dot{\mathcal{K}}_{tip}$ in Eq. (4.11). We can then make similar simplifications in the formula (4.3) for the deviatoric stress. I find

$$s_{\theta\theta}=-s_{\rho\rho}\approx 2\sigma_\infty\sqrt{W}\left(\frac{\eta_0^2}{\eta^3}\right)\left[1-4W\theta^2\left(\frac{2}{\eta^2}-\frac{1}{\eta_0^2}\right)\right] \tag{4.12}$$

and

$$s_{\rho\theta}\approx\frac{4\sigma_\infty\sqrt{W}}{\eta^4}(\eta^2-\eta_0^2)\sqrt{W}\theta, \tag{4.13}$$

where

$$\eta=(\rho^2-m)\sqrt{W} \tag{4.14}$$

and

$$\eta_0=\eta(\rho=1)=\sqrt{\frac{2}{\mathcal{K}_{tip}}} \tag{4.15}$$

is the value of η at the crack surface.

The pair of equations (4.12) and (4.13) is the analog of Eq. (3.5) for the elliptical crack tip. The next step is to deduce a useful analog of the rate equation (3.6). To do this, start with the expressions for the rate-of-deformation tensor D in terms of the material velocity components v_ρ and v_θ [22]:

$$D_{\rho\rho}=\frac{1}{WN}\left(\frac{\partial v_\rho}{\partial\rho}+\frac{v_\theta}{\rho}\frac{1}{N}\frac{\partial N}{\partial\theta}\right), \tag{4.16}$$

$$D_{\theta\theta}=\frac{1}{WN\rho}\left(\frac{\partial v_\theta}{\partial\theta}+\frac{v_\rho}{\rho}\frac{1}{N}\frac{\partial}{\partial\rho}(\rho N)\right), \tag{4.17}$$

and

$$D_{\rho\theta}=\frac{1}{2WN}\left(\frac{1}{\rho}\frac{\partial v_\rho}{\partial\theta}+\frac{\partial v_\theta}{\partial\rho}-\frac{v_\theta}{\rho}-\frac{v_\rho}{N}\frac{\partial N}{\partial\theta}-\frac{v_\theta}{N}\frac{\partial N}{\partial\rho}\right), \tag{4.18}$$

where

$$N^2(\rho,\theta)=1+\frac{m^2}{\rho^4}-\frac{2m}{\rho^2}\cos 2\theta \tag{4.19}$$

is the same quantity that occurs in the metric equation (4.10).

Dimensional analysis plus symmetry about $\theta=0$ implies that we can write v_ρ and v_θ in the form

$$v_\rho(\rho,\theta)=\tilde{v}_\rho(\eta,\theta)\approx\alpha(\eta)\left[1+W\theta^2\left(\frac{a}{\eta^2}+\frac{b}{\eta_0^2}\right)\right], \tag{4.20}$$

$$v_\theta(\rho,\theta)\approx\beta(\eta)\sqrt{W}\theta, \tag{4.21}$$

where the functions $\alpha(\eta)$ and $\beta(\eta)$, and the numerical constants a and b, are to be determined. With these definitions and in the large-W limit,

$$D_{\rho\rho}\approx\frac{2}{\eta}\left(1-\frac{2W\theta^2}{\eta^2}\right)\left(\frac{\partial\tilde{v}_\rho}{\partial\eta}+\frac{2\beta W\theta^2}{\eta^2}\right), \tag{4.22}$$

$$D_{\theta\theta}\approx\frac{1}{\eta}\left(1-\frac{2W\theta^2}{\eta^2}\right)\left[\beta+\frac{2\tilde{v}_\rho}{\eta}\left(1-\frac{4W\theta^2}{\eta^2}\right)\right], \tag{4.23}$$

and

$$D_{\rho\theta}\approx\frac{1}{\eta}\left[\frac{1}{2\sqrt{W}}\frac{\partial\tilde{v}_\rho}{\partial\theta}+\left(\frac{\partial\beta}{\partial\eta}-\frac{1}{\eta^2}(2\alpha+\beta\eta)\right)\theta\sqrt{W}\right]. \tag{4.24}$$

V. ELLIPTICAL APPROXIMATION FOR CRACK-TIP DYNAMICS: APPLICATIONS

As a first exercise in the application of these formulas, consider the case of linear viscoplasticity with a rate-of-deformation tensor D equal to $(\lambda/\tau)s$ and vanishing yield stress. For this special situation, compatibility is automatically satisfied. If we have an elliptical hole at some moment, we can calculate its instantaneous growth rate exactly. (This is not to say that the hole necessarily remains elliptical at later times.) At $\theta=0$,

$$D_{\rho\rho}\approx\frac{2}{\eta}\frac{d\alpha}{d\eta}=-\frac{2\lambda}{\tau}\sigma_\infty\sqrt{W}\frac{\eta_0^2}{\eta^3} \tag{5.1}$$

and

$$D_{\theta\theta}\approx\frac{1}{\eta^2}(2\alpha+\beta\eta)=+\frac{2\lambda}{\tau}\sigma_\infty\sqrt{W}\frac{\eta_0^2}{\eta^3}. \tag{5.2}$$

In both of these equations, the first forms of D come from Eqs. (4.22) and (4.23), i.e., from geometry, and the second from our viscoplastic constitutive relation and the Muskhelishvili solutions for the stress in Eq. (4.12). It follows from Eq. (5.1) that

$$\alpha(\eta)=\frac{\lambda}{\tau}\sigma_\infty\sqrt{W}\,\frac{\eta_0^2}{\eta}, \tag{5.3}$$

and then from Eq. (5.2) that $\beta(\eta)=0$. To compute the constants a and b, use the expressions (4.24) and (4.13) for the off-diagonal element of D:

$$D_{\rho\theta}\approx\frac{\theta\sqrt{W}}{\eta}\alpha(\eta)\left(\frac{a-2}{\eta^2}+\frac{b}{\eta_0^2}\right)$$
$$=\frac{4\lambda}{\tau}\,\frac{\sigma_\infty\sqrt{W}}{\eta^4}(\eta^2-\eta_0^2)\,\theta\sqrt{W}. \tag{5.4}$$

Equating coefficients of η^{-2} and η^{-4}, we have $a=-2$ and $b=4$. Thus,

$$\tilde{v}_\rho(\eta,\theta)=\frac{\lambda}{\tau}\sigma_\infty\sqrt{W}\,\frac{\eta_0^2}{\eta}\left[1+2W\theta^2\left(\frac{2}{\eta_0^2}-\frac{1}{\eta^2}\right)\right]. \tag{5.5}$$

To check compatibility, we can use Eq. (5.5) to evaluate $D_{\rho\rho}$ in Eq. (4.22) and confirm that we recover the fully θ-dependent expression for $(\lambda/\tau)s_{\rho\rho}$ as given by Eq. (4.12).

Setting $\eta=\eta_0$ in Eq. (5.5), we have

$$v_n(\theta)=\frac{\lambda}{\tau}\sigma_\infty\sqrt{\frac{2W}{\mathcal{K}_{tip}}}(1+\mathcal{K}_{tip}W\theta^2), \tag{5.6}$$

from which we find

$$v_{tip}=\frac{\lambda}{\tau}\sigma_\infty\sqrt{\frac{2W}{\mathcal{K}_{tip}}} \tag{5.7}$$

and, using Eq. (4.11),

$$\dot{\mathcal{K}}_{tip}=-\frac{2\lambda}{\tau}\sigma_\infty\sqrt{2W}\mathcal{K}_{tip}^{3/2}. \tag{5.8}$$

Apart from numerical factors, both of these results can be obtained just from dimensional analysis. The parameter λ has the dimensions of inverse stress. The applied stress can occur only in the combination $\sigma_\infty\sqrt{W}$; W cannot appear otherwise, and $\mathcal{K}^{-1}$ is the only other length scale in the problem. Thus, the right-hand side of Eq. (5.7) is the only group of parameters that can have the dimensions of velocity. A similar argument yields Eq. (5.8).

This model, with no yield stress, clearly is describing a ductile material. Equations (5.7) and (5.8) tell us that v_{tip} increases and $\mathcal{K}_{tip}$ decreases with time, which means that the crack tip blunts until the sharp-tip approximation ($\mathcal{K}_{tip}W \gg 1$) becomes invalid. The long, thin ellipse grows out to become an expanding circle. (See [23] for a stability analysis of the growing circular hole.)

Equations (5.7) and (5.8) give us the tip velocity and rate of change of the curvature at some instant of time for specified values of $\mathcal{K}_{tip}$ and W. We also know that $v_{tip}\approx 2\dot{W}$. For present purposes, and especially in the next applications where we shall find steady-state crack motion, it is useful to assume that the quantity $\sigma_\infty\sqrt{W}$ remains constant, that is, that σ_∞ varies in such a way as to determine a fixed value of the stress-intensity factor.

It is especially interesting that, because $\beta(\eta)=0$ in Eq. (5.2), the angular velocity v_θ vanishes near the crack tip. The plastic flow, at least in this particular case, is purely radial. A related feature of these results is that Eq. (5.6) can be written in the form

$$v_n(\theta)\cong\frac{\lambda}{\tau}\,\frac{s_{\theta\theta}(1,\theta)}{\mathcal{K}(\theta)}, \tag{5.9}$$

where $\mathcal{K}(\theta)$, the curvature of the elliptical crack surface, is

$$\mathcal{K}(\theta)=\frac{1-m^2}{WN^{3/2}(1,\theta)}\approx\mathcal{K}_{tip}(1-3\mathcal{K}_{tip}W\theta^2). \tag{5.10}$$

That is, we obtain a correct expression for the normal velocity $v_n(\theta)$ if we use the formula (3.6) for rate of deformation in the circle problem and simply replace the strain rate $\dot{\varepsilon}^{pl}_{\phi\phi}$ by $D_{\theta\theta}(1,\theta)$ and the radius R by the local radius of curvature $\mathcal{K}^{-1}(\theta)$.

It follows that a plausible generalization of Eq. (5.9) for nonzero yield stress, the analog of Eqs. (3.6)–(3.8), is

$$v_n(\theta)\cong\begin{cases}(\lambda/\tau)\mathcal{K}^{-1}(\theta)[s_{\theta\theta}(1,\theta)-s_y] & \text{for } s_{\theta\theta}>s_y,\\ 0 & \text{otherwise,}\end{cases} \tag{5.11}$$

and therefore, for $s_{\theta\theta}(1,\theta)>s_y$,

$$v_n(\theta)\approx\frac{\lambda}{\tau}\sigma_\infty\sqrt{\frac{2W}{\mathcal{K}_{tip}}}\left[\left(1-\frac{s_y}{\sigma_\infty\sqrt{2W\mathcal{K}_{tip}}}\right)\right.$$
$$\left.+\mathcal{K}_{tip}W\theta^2\left(1-\frac{3s_y}{\sigma_\infty\sqrt{2W\mathcal{K}_{tip}}}\right)\right]. \tag{5.12}$$

For $\mathcal{K}_{tip}>s_y^2/(2W\sigma_\infty^2)$, that is, $s_{\theta\theta}(1,0)>s_y$,

$$v_{tip}\cong\frac{\lambda}{\tau}\left(\sigma_\infty\sqrt{\frac{2W}{\mathcal{K}_{tip}}}-\frac{s_y}{\mathcal{K}_{tip}}\right), \tag{5.13}$$

and, with Eq. (4.11),

$$\dot{\mathcal{K}}_{tip}\cong\frac{2\lambda}{\tau}(-\sigma_\infty\sqrt{2W}\mathcal{K}_{tip}^{3/2}+2s_y\mathcal{K}_{tip}). \tag{5.14}$$

The presence of a nonzero s_y completely changes the nature of these results from what we found in Eqs. (5.7) and (5.8). Equation (5.14) has a stable fixed point at a nonzero value of the tip curvature, say, at $\mathcal{K}_{tip}=\mathcal{K}^*_{tip}$, where

$$\mathcal{K}^*_{tip}=\frac{2s_y^2}{W\sigma_\infty^2}. \tag{5.15}$$

[Note that $\mathcal{K}^*_{tip}$ is larger than the minimum value of $\mathcal{K}_{tip}$ consistent with $s_{\theta\theta}(1,\theta)>s_y$.] The tip blunts—$\mathcal{K}_{tip}$

decreases—when $\mathcal{K}_{tip} > \mathcal{K}^*_{tip}$, and it sharpens—$\mathcal{K}_{tip}$ increases—when $\mathcal{K}_{tip} < \mathcal{K}^*_{tip}$. The steady-state tip speed is therefore

$$v^*_{tip} = v_{tip}(\mathcal{K}^*_{tip}) \cong \frac{\lambda}{\tau}\,\frac{\sigma_\infty^2 W}{2 s_y}. \tag{5.16}$$

As advertised in the Introduction, the tip-stress puzzle has disappeared. In the absence of surface tension, the threshold for crack advance is at $\sigma_\infty = 0$. Above this threshold, v_{tip} rises linearly as a function of $\sigma_\infty^2 W$, a quantity that is proportional to the energy release rate G in the purely elastic case. This is brittle behavior. For any nonzero yield stress, plastic deformation is localized near the tip, and the crack finds a stable shape at which it advances steadily.

Moreover, as the driving force and the tip speed increase, the crack becomes blunter according to Eq. (5.15). This is a natural feature of the crack-shape dynamics in which the curvature $\mathcal{K}_{tip}$ controls the stress concentration at the tip. At larger σ_∞, less concentration is needed in order for the tip stress to reach values in the neighborhood of s_y, and the curvature decreases. It is interesting to speculate that this dynamic blunting might lead to a branching instability at large crack speeds—a possibility that is well beyond the range of the present analysis.

The simplicity of the approximation (5.11) makes it possible to include surface tension in this analysis with only a little extra effort. In principle, we need to modify Muskhelishvili's calculation so that, instead of setting the normal stress $\sigma_{\rho\rho}$ equal to zero at the surface of the ellipse, we use

$$\sigma_{\rho\rho}(1,\theta) = \gamma \mathcal{K}(\theta), \tag{5.17}$$

where γ is the surface tension. In practice, this is a major project. I believe that the problem is analytically solvable, and hope to report results in a subsequent publication. For present purposes, however, we do not need so detailed an analysis.

Because Muskhelishvili's calculation uses only linear elasticity, the resulting stress field is the sum of two terms, one proportional to σ_∞ as shown in Eqs. (4.2) and (4.3), and a second proportional to γ. The new term is an additive contribution to $s_{\theta\theta}(1,\theta)$ in Eq. (5.11). Dimensional analysis tells us that the new expressions for v_{tip} and $\dot{\mathcal{K}}_{tip}$ must have the form

$$v_{tip} \cong \frac{\lambda}{\tau}\left(\sigma_\infty \sqrt{\frac{2W}{\mathcal{K}_{tip}}} - \frac{s_y}{\mathcal{K}_{tip}} - c\,\gamma\right) \tag{5.18}$$

and

$$\dot{\mathcal{K}}_{tip} \cong \frac{2\lambda}{\tau}\left(-\sigma_\infty \sqrt{2W}\mathcal{K}_{tip}^{3/2} + 2 s_y \mathcal{K}_{tip} - d\,\gamma \mathcal{K}_{tip}^2\right), \tag{5.19}$$

where c and d are purely numerical coefficients. In the case of linear viscoplasiticity with $s_y = 0$, the stress field including surface tension would still be fully compatible; therefore these results would be exactly correct and the values of c and d would emerge from a calculation analogous to that which led to Eq. (5.5). When s_y is nonzero, the dimensional argument is valid only within the approximation (5.11).

To see what these equations mean, define a dimensionless parameter k^* at the dynamical fixed point:

$$k^* = \frac{\gamma}{\sigma_\infty}\sqrt{\frac{\mathcal{K}^*_{tip}}{2W}}. \tag{5.20}$$

Setting the right-hand side of Eq. (5.19) to zero, we find

$$k^* = k^*(g) = \frac{1}{2d}\left(\sqrt{1 + \frac{4d}{g}} - 1\right), \tag{5.21}$$

and, according to Eq. (5.18), the steady-state crack speed is

$$v^*_{tip} = \frac{\lambda\,\gamma}{\tau}\left(\frac{1}{k^*} - c - \frac{1}{2 g k^{*2}}\right). \tag{5.22}$$

Here

$$g = \frac{W\sigma_\infty^2}{\gamma s_y} \tag{5.23}$$

is a dimensionless group of parameters that is proportional to the energy release rate G. The combination of Eqs. (5.21) and (5.22) tells us that $\tau v^*_{tip}/\gamma\lambda$ is a universal function of g that rises linearly from zero at some threshold value of g (a numerical constant) and, consistent with Eq. (5.16), approaches $g/2$ for large g. For example, if $c = 0.5$ and $d = 1$, the result is almost indistinguishable from the straight line $\tau v^*_{tip}/\gamma\lambda \cong (g - 1.3)/2$. Apart from the nonzero threshold, the surface tension produces no qualitative changes, in particular, no transition from brittle to ductile behavior except at $s_y = 0$. Note also that, at threshold, $\mathcal{K}^*_{tip} \sim s_y/\gamma$ and, with Eq. (4.12), $s_{\theta\theta}(1,0) \sim s_y$. As expected, $\mathcal{K}_{tip}$ has adjusted dynamically so that the concentrated stress at the tip is proportional to the plastic yield stress.

VI. DISCUSSION

I conclude with some remarks about the nature of the STZ theory and how it may relate to fracture dynamics.

The first point to emphasize is the uncertainty of the approximation (5.11). Although the analogous approximation seems roughly accurate in the circle problem for some range of values of λs_y near unity, it could be entirely incorrect for the crack tip. So far as I can see, the only way to test this approximation and the conclusions I draw from it is by numerical analysis. Such a project is high on my priority list.

A different kind of uncertainty is whether the STZ version of plasticity theory is necessary for the picture of crack-tip dynamics presented here or whether qualitatively the same behavior and the same solution of the tip-stress puzzle might be obtained with conventional theories. I needed the STZ model as justification for the approximation (5.11), without which I could not make progress analytically. I suspect, however, that the physics of the STZ model is playing more than just a technically convenient role. As discussed in Sec. III, one important difference between the STZ model and conventional theories is that it predicts a smooth transition from viscoplasticity inside the plastic zone to viscoelasticity outside. Compatibility therefore does not require so large and rigid a plastic zone surrounding the crack tip as is predicted by conventional theories. The comparatively smooth and thin plastic zone in the STZ model is controlled by the quantity λs_y, which may be of order unity, as opposed to the ordinarily very small ratio s_y/μ in conventional theories. The approximation (5.11) makes sense only if we can take the limit $s_y/\mu \to 0$ while keeping nonzero values of the strain-hardening parameter λ.

A related issue has to do with energy balance. In the Griffith analysis [19], the stored elastic energy per unit length of the sample is of order $G^{el} \sim \sigma_\infty^2 L/\mu$, where L is some macroscopic length, say, the width of a very long strip along whose centerline the crack is advancing. (In our elliptical calculations with infinitely distant boundaries in all directions, the quantity W plays the role of this macroscopic length.) The stored elastic energy vanishes in the limit $\mu \to \infty$, but that limit is permissible only because there are other degrees of freedom in the system that account for displacements of the material. In particular, if the STZ viscoelastic law (2.3) remains valid down to arbitrarily small stress, then the work done in loading the strip is proportional to $G^{pl} \sim \lambda \sigma_\infty^2 L$. This is much bigger than G^{el} if, as we have assumed, $\mu\lambda \gg 1$. Note that this assumption is exactly opposite to that of Freund and Hutchinson [8], whose analysis was limited to situations in which $G^{el} \gg G^{pl}$.

There remains an interesting issue here. In conventional interpretations, G^{pl} is nonrecoverable energy, unavailable for creating new fracture surfaces. That is not necessarily the case for the STZ model. Consider the example of a solid expanding uniformly under negative pressure, perhaps the plate discussed in Sec. III without the hole in it. In addition to purely elastic expansion, there may be vacancy formation, or vacancies may diffuse in from the surface. This is a kind of irreversible bulk plasticity; the vacancies do not instantaneously disappear when the system is unloaded, and the plate does not immediately recover its initial shape. However, as the system comes to equilibrium, the vacancies may find it energetically and kinetically favorable to coalesce and form voids, so that part of their stored energy is converted to new surface energy. I suspect that the shear transformation zones are playing a similar role—that they are created or reoriented in the deforming region ahead of the crack tip, and that part of the plastic work done in this process is converted to surface energy as the crack advances.

This picture becomes even more interesting and complex in the full STZ theory [14], where a strongly stress-dependent rate factor produces hysteretic effects. Perhaps the most important advantage of the STZ theory, one which we have not exploited here, is that plastically deformed regions of a material are characterized not just by displacement fields but also by the state variable Δ. The rate factor in the full theory effectively vanishes at small stress; the material creeps on extremely long time scales. Thus, Δ remains nearly zero when a previously undeformed material is subjected to a small stress. On the other hand, when a strongly deformed material is unloaded, Δ remains at whatever value it reached during the deformation. That residual value of Δ is the means by which the system ''remembers'' its history; it determines how the system responds to subsequent loadings. In fracture, the hardened material left along the crack, in the wake of the plastic zone, will be described by nonzero values of Δ.

This strongly stress-dependent rate factor in the STZ theory has another implication for fracture analysis. The viscoelastic law (2.3) is not strictly valid far ahead of the crack tip unless the system is everywhere so highly stressed or given so long an equilibration time that the missing rate factor is unimportant. In realistic situations, therefore, energy is stored elastically in distant regions where the stress is small enough that the viscoelastic response cuts off. As usual, it is the elastic energy that ultimately drives fracture. In cases where the plastic dissipation rate is comparable to or smaller than the bare fracture energy, the behavior is likely to be highly sensitive to details of the plastic constitutive laws. On the other hand, if $\lambda\mu \gg 1$ and plastic dissipation is dominant, then some generalization of the infinite-μ calculation presented here should be accurate.

Finally, there are two basic questions to which I have alluded only briefly so far. First, what has happened to the brittle-ductile transition, which we have seen here only at $s_y = 0$? The mathematical signature of ductility in this theory would be the disappearance of the fixed point curvature $\mathcal{K}_{tip}^*$ so that, as in Eq. (5.8), the tip blunts indefinitely. That might happen for some or all values of the driving force, for nonzero s_y. One possibility is that such a mechanism has been lost here in the approximation (5.11). Another is that the extreme stiffness implied by the $\mu \to \infty$ limit suppresses ductility. Both possibilities may simultaneously be correct.

Second, there is the question of whether or when this picture of crack-tip dynamics governed by plasticity might be valid. At first glance, we might guess that the picture is plausible only for highly deformable materials with small s_y where plastic flow in the neighborhood of a crack tip might resemble fluid flow near a viscous finger [24]. I suggest that the picture is much more generally correct, at least as long as I am allowed to adopt a liberal interpretation of the term ''plastic deformation.''

Falk's recent molecular dynamics simulations [25] of brittle fracture in amorphous solids clearly show crack tips that are blunt on scales of roughly ten atomic spacings and significant amounts of STZ activity near these tips. Even the simulations of Zhou *et al.* [26] and of Marder and coworkers [27] show molecular rearrangements and blunting on the scale of a few atomic spacings at the tips of cracks in defect-free crystalline solids. I do not suggest, for either of these cases, that the molecular rearrangements near the crack tip are accurately described by any continuum theory of plasticity. Nevertheless, the idea that dynamic blunting brings the stress at the tip down to values of order s_y seems to make sense, whether or not the continuum approximation is quantitatively correct.

As pointed out at the end of Sec. V, near threshold, $\mathcal{K}_{tip}^*$ is proportional to s_y/γ, a quantity that is small in easily deformable materials where s_y is small. On the other hand, if we take the Dugdale assumption literally, then s_y is the cohesive stress, and γ is equal to s_y times the range of the cohesive forces. In this case, $\mathcal{K}_{tip}^*$ is of the order of inverse atomic spacings. While the continuum approximation cannot be strictly correct at such small length scales, its prediction is consistent with our physical picture of the characteristic sharpness of crack tips. We may see here a way to bridge the gap between the present picture of fracture in deformable materials and Marder's suggestion [28] that, in rigid single crystals, the length scale at which stress is regularized is simply the atomic spacing.

In short, I argue that the tip-stress puzzle becomes a non-issue if we include the tip curvature as a relevant degree of freedom near the crack tip. It follows, I believe, that this degree of freedom is an essential ingredient in theories of dynamic fracture.

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